Consistent Linearization of Euler's *Elastica*

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The following is the equation governing the transverse displacement u of an initially straight, planar Euler-Bernoulli beam of linear density ρ_0 and flexural rigidity EI, subjected to a compressive axial load P:

$$\rho_0 \ddot{u} + (EIu'')'' + Pu'' = 0\,. \tag{1}$$

There are many ways to derive Eq. (1), most involving an integration of the equations of linear elasticity through the thickness of the beam. These derivations range in sophistication from the *ad hoc* approach taught in courses on the mechanics of materials, to mathematical techniques using Γ-convergence.

Not often discussed is the fact that Eq. (1) is also the result of a consistent linearization of the equations governing Euler's (inextensible) *elastica*, which in their most condensed form are

$$\rho_0 \ddot{\mathbf{r}} + (EI\mathbf{r}'')'' = (\sigma \mathbf{r}')' \quad \text{and} \quad \mathbf{r}' \cdot \mathbf{r}' = 1\,. \tag{2}$$

Here, $\mathbf{r}(\xi, t)$ is the position vector of the material point $\xi \in [0, L]$ at time t relative to some fixed origin. $\sigma(\xi, t)$ is a Lagrange multiplier that enforces the constraint of inextensibility, and should not be confused with the tension. $\mathbf{r}$ and σ are the only unknowns in this formulation, but useful for prescribing boundary conditions is the following expression for the internal force:

$$\mathbf{n} = \sigma \mathbf{r}' - (EI\mathbf{r}'')'\,. \tag{3}$$

We define constant unit vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively parallel and perpendicular to the undeformed rod, so that the displacement is

$$\mathbf{u}(\xi, t) = \mathbf{r}(\xi, t) - \xi \mathbf{E}_1\,, \tag{4}$$

We have made no assertion that the displacement be purely transverse. With Eq. (4) in mind, Eq. (2) becomes

$$\rho_0 \ddot{\mathbf{u}} + (EI\mathbf{u}'')'' = [\sigma(\mathbf{E}_1 + \mathbf{u}')]' \quad \text{and} \quad \mathbf{u}' \cdot \mathbf{u}' + 2\mathbf{u}' \cdot \mathbf{E}_1 = 0\,. \tag{5}$$

Our goal is to linearize Eq. (5) about configurations with $\mathbf{u}(\xi, t) = \mathbf{0}$ but $\sigma(\xi, t)$ not necessarily zero, so we set $\mathbf{u}(\xi, t) = \epsilon \bar{\mathbf{u}}(\xi, t)$ and insert it into Eq. $(5)_2$. Dividing the result by ϵ and letting $\epsilon \to 0$, we find $\bar{\mathbf{u}}' \cdot \mathbf{E}_1 = 0$. In other words, only rigid-body displacements are possible in the direction of $\mathbf{E}_1$. Likewise, Eq. $(5)_1$ becomes

$$\epsilon \Big[\rho_0 \ddot{\bar{\mathbf{u}}} + (EI\bar{\mathbf{u}}'')''\Big] = \sigma' \mathbf{E}_1 + \epsilon(\sigma' \bar{\mathbf{u}}' + \sigma \bar{\mathbf{u}}'')\,. \tag{6}$$

Taking the dot product of Eq. (6) with $\mathbf{E}_1$ yields $\sigma' = 0$, or equivalently $\sigma = \sigma(t)$. Taking the dot product with $\mathbf{E}_2$ and dividing by ϵ gives

$$\rho_0 \ddot{u} + (EIu'')'' - \sigma u'' = 0\,, \tag{7}$$

where $u = \bar{\mathbf{u}} \cdot \mathbf{E}_2$. Upon inserting Eq. (4) into Eq. (3) and writing $\mathbf{u} = \epsilon \bar{\mathbf{u}}$ once more, we see that

$$\mathbf{n} = \sigma \mathbf{E}_1 + \epsilon \Big[\sigma \bar{\mathbf{u}}' - (EI\bar{\mathbf{u}}'')'\Big] \to \sigma \mathbf{E}_1 \quad \text{as} \quad \epsilon \to 0\,, \tag{8}$$

thus making clear that σ is in fact the internal tension in the limit of small deflections. Equations (1) and (7) are identical upon identifying that $\sigma = -P$.